Econ 802
Answers to Second Midterm

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1(a) Average cost is $\frac{c(y)}{y}$. Differentiate this to get

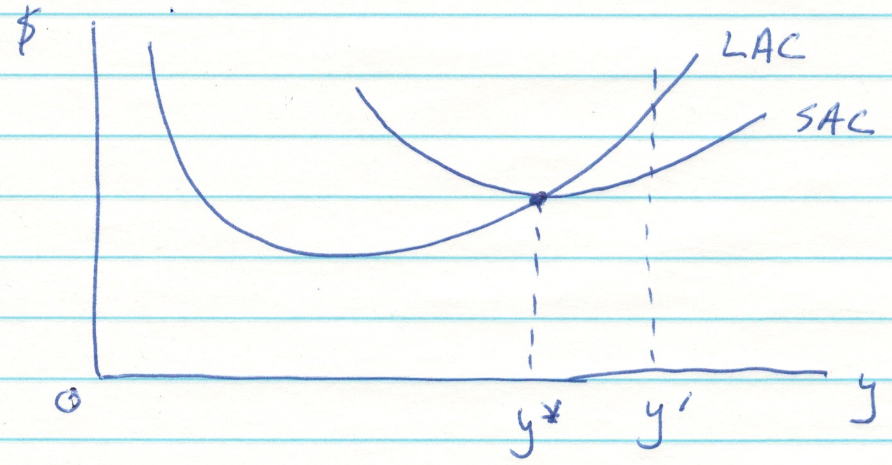
$$\frac{d\left[\frac{c(y)}{y}\right]}{dy} = \frac{c'(y)}{y} - \frac{c(y)}{y^2} = \frac{1}{y}\left[c'(y) - \frac{c(y)}{y}\right]$$

where $c'(y)$ is marginal cost.

If AC is falling we must have $c'(y) < \frac{c(y)}{y}$

If AC is rising we must have $c'(y) > \frac{c(y)}{y}$

(b) We use a proof by contradiction. Suppose the LAC curve does pass through the minimum point of the SAC curve. Then we get a graph like this:



There must be some $y' > y^*$ such that $LAC(y') > SAC(y')$ or $\frac{c(w,y')}{y'} > \frac{c(w,y',x_n)}{y'}$, which implies $c(w,y') > c(w,y',x_n)$.

But this is impossible because the minimum cost of y' can never be higher in the long run than in the short run (in the LR, the firm could always do the same thing it does in the SR).

1 (c) The firm minimizes $w \cdot x$ subject to $f(x) = y$.

The Lagrangean is $L = w \cdot x - \lambda[f(x) - y]$

The FOC are $w = \lambda \dfrac{\partial f(x)}{\partial x}$ and $f(x) = y$

Let the solution for the FOC be $x(y)$ and $\lambda(y)$

(we ignore w because these prices are held constant)

Write the LR cost function as

$$c(w,y) \equiv w \cdot x(y) - \lambda(y)[f(x(y)) - y]$$

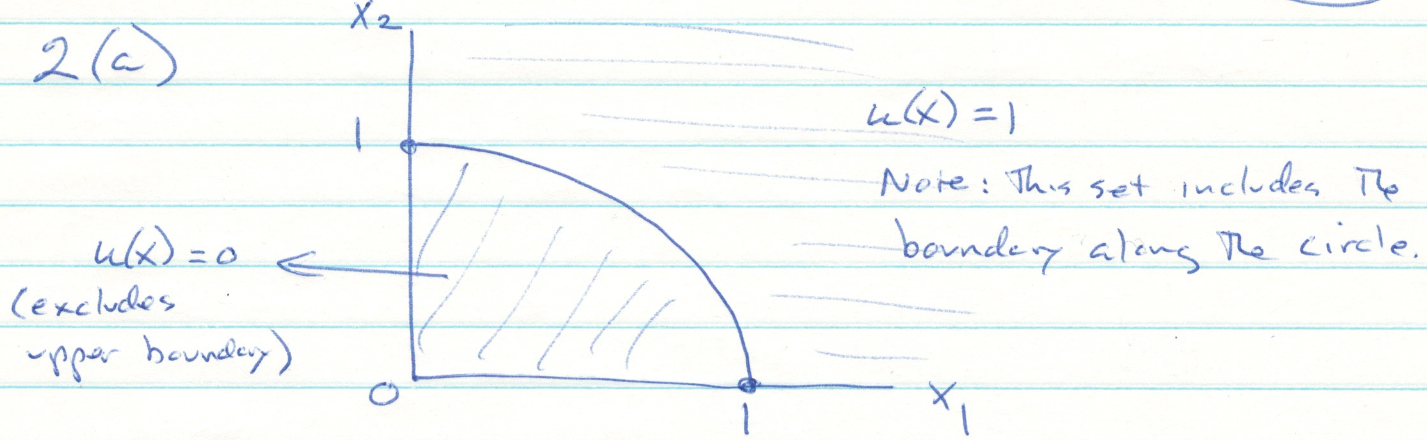
Now differentiate with respect to y. This gives

$$\frac{\partial c(w,y)}{\partial y} = w \cdot \frac{\partial x}{\partial y} - \lambda \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} - \lambda'[f(x(y)) - y] + \lambda(y)$$

$$= \underbrace{\left[w - \lambda \frac{\partial f}{\partial x}\right]}_{\substack{= 0 \text{ at } x(y) \\ \text{due to FOC}}} \frac{\partial x}{\partial y} - \lambda' \underbrace{[f(x(y)) - y]}_{\substack{= 0 \text{ at } x(y) \\ \text{due to FOC}}} + \lambda(y)$$

So $\dfrac{\partial c(w,y)}{\partial y} = \lambda(y)$ where $\dfrac{\partial c(w,y)}{\partial y}$ is LRMC.

2(a)



(i) Weak monotonicity: yes. Increasing x_1, x_2, or both cannot reduce utility and could increase it.

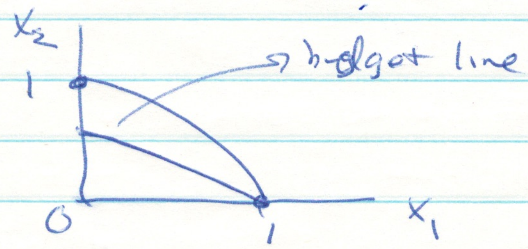
(ii) Strong monotonicity: NO. Starting from a point in the interior of either utility set, adding a small amount of either or both goods will keep utility constant.

(iii) convexity: NO. The set of points with $u(x)=1$ is not a convex set. For example, points on the line segment between $(1,0)$ and $(0,1)$ are in the set $u(x)=0$, not $u(x)=1$.

(iv) strict convexity: NO. The set of points with $u(x)=1$ is not convex, so it cannot be strictly convex.

(v) local non-satiation: NO. Choose any point x in the interior of the set for $u(x)=0$. For sufficiently small $\epsilon > 0$, all points y with $|x-y|<\epsilon$ must also have $u(y)=0$.

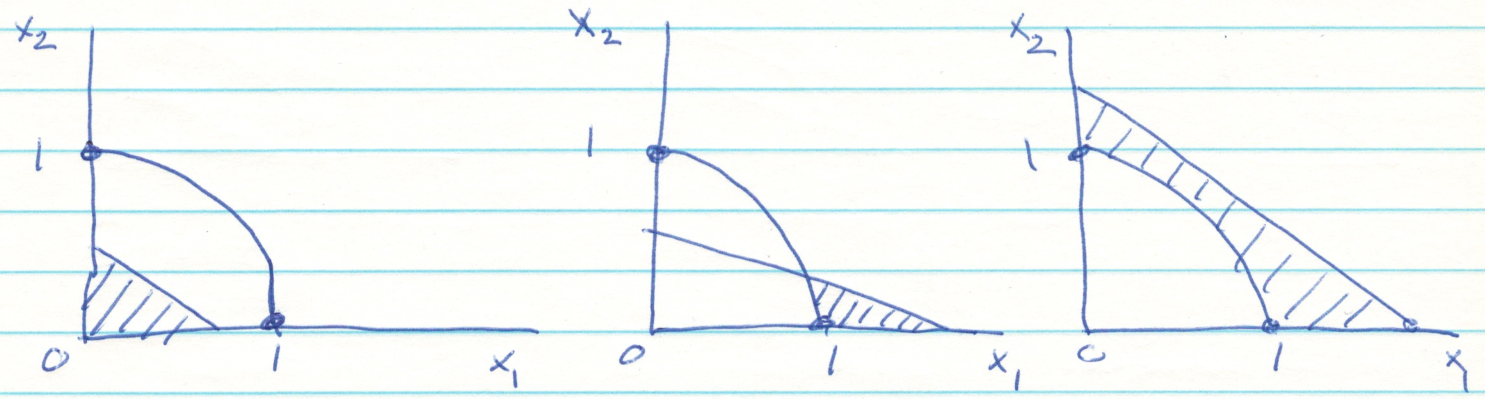
(b) Unique solution:



$p_1 < p_2$ implies that the budget line is flatter than -1. If Joe has just enough income to afford $x^* = (1,0)$ then this is the unique solution, because $u(1,0)=1$ and all other feasible bundles have $u(x)=0$.

2(b) continued:

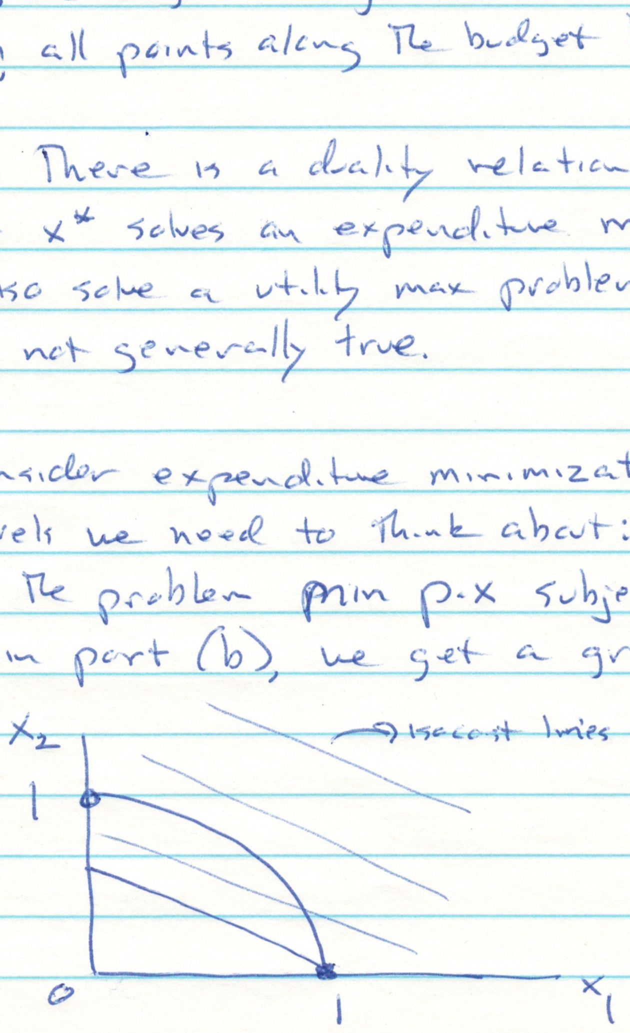
Many solutions: This can occur in various ways (you don't need to discuss all 3 cases, one example is enough).



In each case, all shaded points are optimal. In the left graph, all feasible points have $u(x)=0$. In the middle graph, some points along the budget line give $u(x)=1$ while others do not. In the right graph, all points along the budget line give $u(x)=1$.

(c) There is a duality relationship in a limited sense. If x^* solves an expenditure minimization problem, it will also solve a utility max problem. However, the converse is not generally true.

Consider expenditure minimization. There are only two utility levels we need to think about: $u=1$ and $u=0$. First look at the problem $\min p \cdot x$ subject to $u(x) \geq 1$. If $p_1 < p_2$ as in part (b), we get a graph like this:

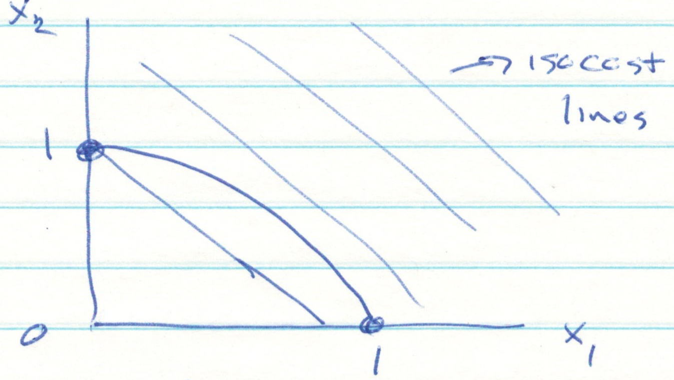


The expenditure min point (unique) is $x^* = (1, 0)$ and the resulting expenditure is p_1. Call this expenditure m^*.

2(c) continued:

It should be clear from the graph that if Joe has income m^*, the point $x^* = (1,0)$ also solves the problem of maximizing utility subject to $px \le m^*$. This gives a duality relationship.

The same reasoning applies when $p_1 > p_2$. In this case, the expenditure-minimizing bundle is $x^* = (0,1)$ and the resulting expenditure is p_2. Call this expenditure m^*. Then $x^* = (0,1)$ also solves the problem of maxing utility subject to $px \le m^*$.



When $p_1 = p_2 > 0$, there are two points that minimize expenditure, $(1,0)$ and $(0,1)$; each will also max utility.

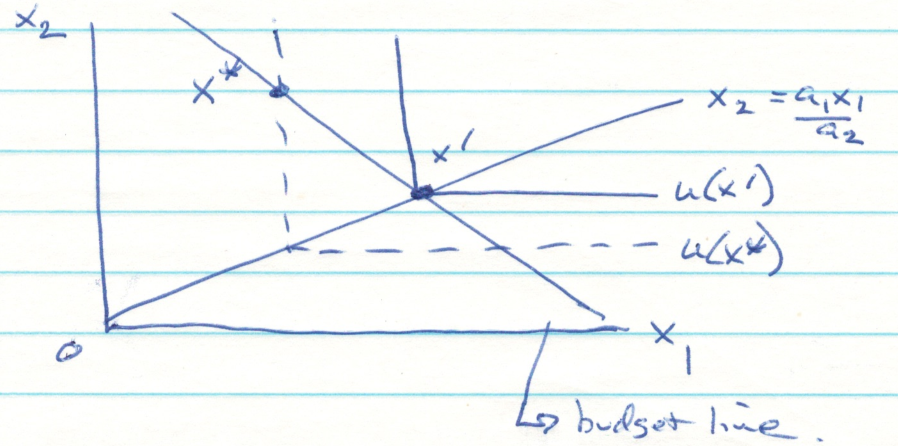
[Note: min px subject to $u(x) \ge 0$ gives $x^* = 0$ and $m^* = 0$. This gives a trivial kind of duality.]

However, problems arise if we pick some arbitrary point x^* that solves the utility max problem for $px \le m$. As shown in the answer to 2(b), there could be many solutions, and in general points on the budget line do not solve ~~add~~ an expenditure min problem. For example, in the middle and right graphs in 2(b), the point that minimizes expenditure is $(1,0)$, which is below the budget line.

The difficulty here arises from the absence of local non-satiation. Utility maximizing points can be on the budget line but they can also be below it.

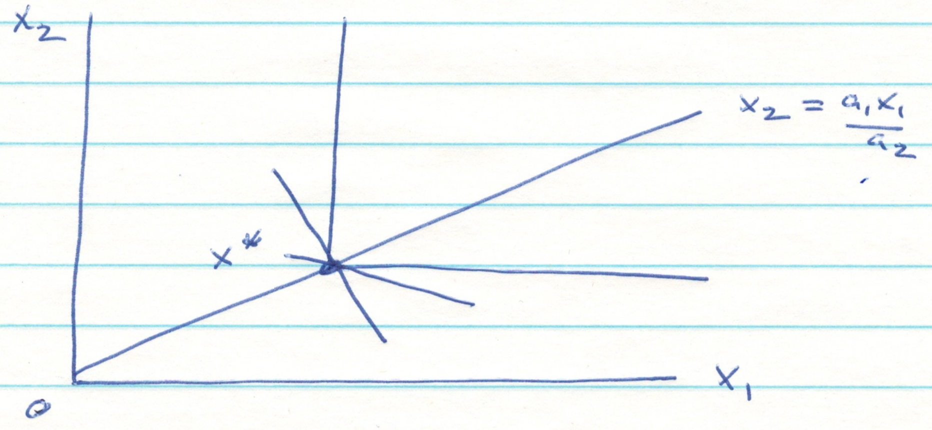
3 (a) Consider the set of points with $u = a_1 x_1 = a_2 x_2$ where $x_2 = \frac{a_1 x_1}{a_2}$. This gives a ray from the origin. There are two possible cases: either x^* is on this ray or not.

(i) If <u>x^* is not on the ray</u> we have something like this:



For any prices $p > 0$, it is possible to move along the budget line and increase utility from $u(x^*)$ to $u(x')$. So there cannot be <u>any</u> prices $p > 0$ such that x^* is optimal.

(ii) If <u>x^* is on the ray</u> we get this:



Now the problem is that <u>any</u> budget line passing through x^* makes x^* optimal (indicated by different slopes of budget lines corresponding to different price vectors p). So there is no <u>unique</u> p^* such that x^* is optimal (the inverse demand function is not well-defined).

3 (b) We need to solve $\min p \cdot x$ subject to $\min\{a_1x_1 \ldots a_nx_n\} \geq u$.
The solution cannot have $a_i x_i > a_j x_j$ for any i, j because if it did, we could reduce x_i slightly and lower expenditure while still achieving the same utility as before. Therefore the solution has $a_1x_1 = a_2x_2 = \ldots = a_nx_n = u$. Thus $x_i = \frac{u}{a_i}$ for all $i \Rightarrow$ the Hicksian demands are

$$h_i(p, u) = \frac{u}{a_i} \quad \text{for all } i = 1 \ldots n$$

(note that these are independent of the prices).
The expenditure function is $e(p, u) = \sum_i p_i h_i(p, u)$
or $e(p, u) = \sum_i p_i \left(\frac{u}{a_i}\right)$
$\Rightarrow \quad e(p, u) = u \left[\sum_i \frac{p_i}{a_i}\right]$

(c) We need to solve $\max \{\min\{a_1x_1 \ldots a_nx_n\}\}$
subject to $\sum_i p_i x_i \leq m$

The solution cannot have $a_i x_i > a_j x_j$ for any i, j.
Suppose it did. Then we could reduce x_i slightly and get the same utility as before, without violating the budget constraint. The money saved by reducing x_i could be used to increase the levels of all other goods, which would raise utility. This is a contradiction.
Therefore the solution must have $a_1x_1 = a_2x_2 = \ldots = a_nx_n$
So $x_2 = \frac{a_1x_1}{a_2}, \ldots, x_n = \frac{a_1x_1}{a_n}$.

Substitute these expressions into the budget constraint $\sum_i p_i x_i = m$ to get $p_1x_1 + p_2\frac{a_1x_1}{a_2} + \ldots p_n\frac{a_1x_1}{a_n} = m$

$$\Rightarrow x_1\left[p_1 + p_2\frac{a_1}{a_2} + \ldots p_n\frac{a_1}{a_n}\right] = m$$

3 (c) continued
$$\Rightarrow x_1 = \frac{m}{\left[p_1 + p_2 \frac{a_1}{a_2} + \dots p_n \frac{a_1}{a_n}\right]}$$

$$\Rightarrow x_1 = \frac{\left(\frac{m}{a_1}\right)}{\sum_j \frac{p_j}{a_j}} \text{; in general, } x_i(p, m) = \frac{\left(\frac{m}{a_i}\right)}{\sum_j \frac{p_j}{a_j}}$$

These are the Marshallian demands. Because $v(p, m) = a_i x_i(p, m)$ for all $i = 1 \dots n$,

we can write $v(p, m) = a_1 x_1(p, m) = \dfrac{m}{\sum_j \frac{p_j}{a_j}}$

This gives the indirect utility function.

4 (a) We have the Marshallian demand $L(p, w, m) = \dfrac{\alpha m}{w}$

The Slutsky equation says
$$\frac{\partial L}{\partial w} = \frac{\partial h_L}{\partial w} - \frac{\partial L}{\partial m} \cdot L$$
where h_L is the Hicksian demand for leisure.

We don't know the Hicksian demand function h_L but we can calculate the other expressions from the Marshallian demand:

Total effect: $\dfrac{\partial L}{\partial w} = -\dfrac{\alpha m}{w^2} < 0$

(negative because leisure is a normal good and Jane is worse off)

Income effect: $-\dfrac{\partial L}{\partial m} \cdot L = -\left(\dfrac{\alpha}{w}\right)\left(\dfrac{\alpha m}{w}\right) = -\dfrac{\alpha^2 m}{w^2} < 0$

The substitution effect and income effect must add up to the total effect, so

Substitution effect: $\dfrac{\partial L}{\partial w} + \dfrac{\partial L}{\partial m} \cdot L = -\dfrac{\alpha m}{w^2} + \dfrac{\alpha^2 m}{w^2}$

$$= -\frac{\alpha m}{w^2}(1 - \alpha) < 0$$

(negative because $w \uparrow$ increases the price of leisure and Hicksian demands slope down)

4(b) Substitute $m = wT + r$ into the Marshallian demand for leisure to get $L(p, w, wT+r) = \frac{\alpha(wT+r)}{w}$.
If $r=0$, this reduces to
$L(p, w, wT) = \frac{\alpha w T}{w} = \alpha T$, which is a constant.
This implies that the wage has no effect on leisure.
The reason why this differs from part (a) is that previously the income m was fixed independently of w. Here the income m depends on w because there is an endowment of time in the budget constraint. When w changes the effect operating through the price of leisure cancels out against the effect operating through the value of Jane's endowment (her income).

(c) The Marshallian demand for leisure and the Marshallian demand for consumption must both satisfy the budget constraint, so

$$pc(p,w,m) + wL(p,w,m) \equiv m$$

$$\Rightarrow \quad pc(p,w,m) + w\left[\frac{\alpha m}{w}\right] \equiv m$$

$$\Rightarrow \quad pc(p,w,m) \equiv m(1-\alpha)$$

$$\Rightarrow \quad c(p,w,m) = \frac{m(1-\alpha)}{p}$$

Substitute $m = wT + r$ to get $c(p,w,r) = \frac{(1-\alpha)(wT+r)}{p}$
Yes, the comparative statics make sense.
If $p \uparrow$ then $c \downarrow$, so there is a downward sloping demand for consumption.
If $w \uparrow$ then $c \uparrow$ because she is richer and consumption is a normal good.
If $r \uparrow$ then $c \uparrow$ because again she is richer, and consumption is a normal good.

5 (a) The first thing we notice is that the utility function is quasi-linear. Therefore we can solve the budget constraint to get $x_2 = \frac{m - p_1 x_1}{p_2}$ and substitute this into the utility function:

$$u = x_1^{1/2} + \frac{m}{p_2} - \frac{p_1 x_1}{p_2}$$

Now we have an unconstrained max problem with just one variable.

FOC: $\frac{1}{2} x_1^{-1/2} - \frac{p_1}{p_2} = 0$

SOC: $-\left(\frac{1}{4}\right) x_1^{-3/2} < 0$ which is sufficient for a max.

Solving the FOC gives $\frac{1}{2} = \frac{p_1}{p_2} x_1^{1/2} \Rightarrow x_1^{1/2} = \frac{p_2}{2p_1}$

$\Rightarrow x_1(p, m) = \left(\frac{p_2}{2p_1}\right)^2$

Note that due to the quasi-linearity, m drops out here. From the budget constraint,

$$x_2(p, m) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1(p, m)$$

$$\Rightarrow x_2(p, m) = \frac{m}{p_2} - \left(\frac{1}{4}\right)\left(\frac{p_2}{p_1}\right)$$

The indirect utility function is $v(p, m) = [x_1(p, m)]^{1/2} + x_2(p, m)$

$$\Rightarrow v(p, m) = \frac{p_2}{2p_1} + \frac{m}{p_2} - \left(\frac{1}{4}\right)\frac{p_2}{p_1}$$

$$\Rightarrow v(p, m) = \frac{p_2}{4p_1} + \frac{m}{p_2}.$$

(b) For the Hicksian demand, we need to minimize $p_1 x_1 + p_2 x_2$ subject to $x_1^{1/2} + x_2 = u$. The constraint gives $x_2 = u - x_1^{1/2}$. Substitute this into the objective to get $p_1 x_1 + p_2 (u - x_1^{1/2})$.

Now we want to minimize $p_1 x_1 + p_2 u - p_2 x_1^{1/2}$

FOC: $p_1 - p_2\left(\frac{1}{2}\right) x_1^{-1/2} = 0$

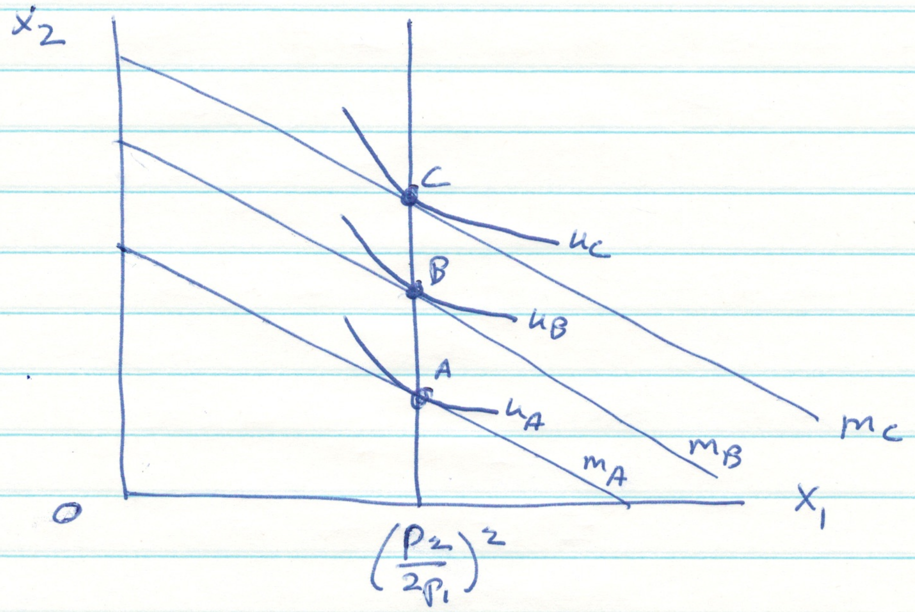
SOC: $p_2\left(\frac{1}{4}\right) x_1^{-3/2} > 0$ which is sufficient for a min.

Solving the FOC gives $p_1 = \frac{p_2}{2} x_1^{-1/2} \Rightarrow x_1^{1/2} = \frac{p_2}{2p_1}$

$\Rightarrow x_1 = \left(\frac{p_2}{2p_1}\right)^2$

This is the Hicksian demand so we write $h_1(p, u) = \left(\frac{p_2}{2p_1}\right)^2$

Note that due to the quasi-linearity, u drops out here.



Because $x_1(p, m)$ is not affected by m, as income rises x_1 does not change. So incomes m_A, m_B, m_C all give the same Marshallian demand $\left(\frac{p_2}{2p_1}\right)^2$. Likewise, $h_1(p, u)$ is not affected by u, so as utility rises x_1 does not change. So the utilities u_A, u_B, u_C all give the same Hicksian demand, which is again $\left(\frac{p_2}{2p_1}\right)^2$. Due to the absence of income effects for good 1 the Marshallian and Hicksian demand functions are identical.

5 (c) Let's generalize the previous utility function so now it is $u(x) = x_1^{\alpha} + x_2$ where $0 < \alpha < 1$.
As in part (a) we have $x_2 = \frac{m}{p_2} - \frac{p_1 x_1}{p_2}$

So max $x_1^{\alpha} + \frac{m}{p_2} - \frac{p_1 x_1}{p_2}$

FOC: $\alpha x_1^{\alpha-1} - \frac{p_1}{p_2} = 0 \Rightarrow x_1^{\alpha-1} = \frac{p_1}{\alpha p_2} \Rightarrow x_1^{1-\alpha} = \frac{\alpha p_2}{p_1}$

$$\Rightarrow x_1(p,m) = \left[\frac{\alpha p_2}{p_1}\right]^{\frac{1}{1-\alpha}}$$

This is the Marshallian demand for good 1. From budget constraint,

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}\left[\frac{\alpha p_2}{p_1}\right]^{\frac{1}{1-\alpha}}$$

So indirect utility is $V(p,m) = \left[\frac{\alpha p_2}{p_1}\right]^{\frac{\alpha}{1-\alpha}} - \frac{p_1}{p_2}\left[\frac{\alpha p_2}{p_1}\right]^{\frac{1}{1-\alpha}} + \frac{m}{p_2}$

Suppose consumer i has the exponent α_i.
We can define

$$a_i(p) = \left[\frac{\alpha_i p_2}{p_1}\right]^{\frac{\alpha_i}{1-\alpha_i}} - \frac{p_1}{p_2}\left[\frac{\alpha_i p_2}{p_1}\right]^{\frac{1}{1-\alpha_i}} \quad \text{(does not involve } m\text{)}$$

and $b(p) = \frac{1}{p_2}$ (which is independent of i)

This allows us to write $V_i(p, m_i) = a_i(p) + b(p) \cdot m_i$ which is in the Gorman form. Therefore we can aggregate across consumers and the market demands $X^1(p, M)$, $X^2(p, M)$ can be treated as if they came from one big consumer with the indirect utility function

$$V(p, M) = \sum_i a_i(p) + \frac{M}{p_2}.$$

(using Roy's identity on this would give the market demands)